Modeling the behavior of longitudinal shear cracks in bi-layer strip

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Let's consider a biflex strip made of two completely bound isotropic bands with elastic properties $\mu_1$ and $\mu_2$ (where $\mu_j$ is the modulus of elasticity). The strip has a tearing mode crack at $y = 0$, $|x| \leq l$. Equal in magnitude but oppositely directed stresses are applied to the edges of the crack (see Figure 1). The first homogeneous isotropic elastic medium occupies the interval $-\infty < x < \infty$, $0 \leq y \leq h_1$, and the second medium occupies the interval $-\infty < x < \infty$, $h_2 \leq y \leq 0$. The biases on the surface of the bands are zero. The physical meaning of this condition is that the surfaces of the biflex strip are based on absolutely rigid bodies. At infinity condition, stresses and biases tend to zero.

The geometry of the investigated problem is shown in Figure 1.

Fig. 1.
In the case of out-of-plane fracture, the nonzero components of the stress tensor \( \sigma_{yz}(x,y) \) and \( \sigma_{xz}(x,y) \) are related to the bias \( w(x,y) \) as follows:

\[
\sigma_{yz}(x,y) = \mu \frac{\partial w(x,y)}{\partial y}, \quad \sigma_{xz}(x,y) = \mu \frac{\partial w(x,y)}{\partial x},
\]

wherein

\[
\frac{\partial^2 w(x,y)}{\partial x^2} + \frac{\partial^2 w(x,y)}{\partial y^2} = 0,
\]

Boundary conditions

\[
|x| < \infty, w_1(x,h_1) = 0, \quad (1)
\]

\[
|x| < \infty, w_2(x,-h_2) = 0, \quad (2)
\]

\[
|x| \leq l, (\sigma_{yz})_1(x,+0) = (\sigma_{yz})_2(x,-0) = -\sigma(x) = -\sigma(-x), \quad (3)
\]

\[
|x| > l : w_1(x,+0) = w_2(x,-0), \quad (4)
\]

\[
|x| > l : (\sigma_{yz})_1(x,+0) = (\sigma_{yz})_2(x,-0) \quad (5)
\]

Conditions at the edges of the crack [2]

\[
\lim_{x \to -l-0} \left\{ \sqrt{2\pi(l-x)} \left[ \frac{\partial w_1(x,+0)}{\partial x} - \frac{\partial w_2(x,-0)}{\partial x} \right] \right\} = - \left( \frac{k+1}{\mu_t} \right) K_{III}. \quad (6)
\]

or

\[
\lim_{x \to l+0} \left\{ \sqrt{2\pi(x-l)} (\sigma_{yz})_1(x,+0) \right\} = -K_{III}, \quad (7)
\]

\[
\lim_{x \to -l-0} \left\{ \sqrt{2\pi(l-x)} \left[ \frac{\partial w_1(x,+0)}{\partial x} - \frac{\partial w_2(x,-0)}{\partial x} \right] \right\} = - \left( \frac{k+1}{\mu_t} \right) K_{III}. \quad (8)
\]

Infinity conditions

\[
|y| < h, |x| \to \infty \{ (\sigma_{yz})_j, (\sigma_{xz})_j \} \to 0, w_j \to 0. \quad (9)
\]

Here, we see that \( w_j(x,y) \) (\( j = 1, 2 \)) are the biases in the first and the second elastic medium; \( (\sigma_{yz})_j(x,y) \) are the stresses in the first and the second elastic medium; \( K_{III} \) is the stress intensity factor at the top of the tearing mode crack; \( \sigma(-x) = -\sigma(x) \).
Applying the Fourier cosine transformation to the Laplace's equation for biases, we get an ordinary differential equation

\[ \frac{d^2 w^*(\lambda, y)}{dy^2} - \lambda^2 w^*(\lambda, y) = 0, \]

where

\[ w_j^*(\lambda, y) = \sqrt{\frac{2}{\pi}} \int_0^\infty w_j(x, y) \cos \lambda x dx, \]

**First elastic medium**

\[ w_1(x, y) = \sqrt{\frac{2}{\pi}} \int_0^\infty A(\lambda) \frac{\text{sh} \lambda (h_1 - y)}{\text{sh} \lambda h_1} \cos \lambda x d\lambda, \quad (9) \]

\[ \left( \sigma_{yz} \right)_1(x, y) = -\mu_1 \sqrt{\frac{2}{\pi}} \int_0^\infty \lambda A(\lambda) \frac{\text{ch} \lambda (h_1 - y)}{\text{sh} \lambda h_1} \cos \lambda x d\lambda, \quad (10) \]

\[ \left( \sigma_{xz} \right)_1(x, y) = -\mu_1 \sqrt{\frac{2}{\pi}} \int_0^\infty \lambda A(\lambda) \frac{\text{sh} \lambda (h_1 - y)}{\text{sh} \lambda h_1} \sin \lambda x d\lambda. \quad (11) \]

**Second elastic medium**

\[ w_2(x, y) = \sqrt{\frac{2}{\pi}} \int_0^\infty C(\lambda) \frac{\text{sh} \lambda (h_2 + y)}{\text{sh} \lambda h_2} \cos \lambda x d\lambda, \quad (12) \]

\[ \left( \sigma_{yz} \right)_2(x, y) = -\mu_2 \sqrt{\frac{2}{\pi}} \int_0^\infty \lambda C(\lambda) \frac{\text{ch} \lambda (h_2 + y)}{\text{sh} \lambda h_2} \cos \lambda x d\lambda, \quad (13) \]

\[ \left( \sigma_{xz} \right)_2(x, y) = -\mu_2 \sqrt{\frac{2}{\pi}} \int_0^\infty \lambda C(\lambda) \frac{\text{sh} \lambda (h_2 + y)}{\text{sh} \lambda h_2} \sin \lambda x d\lambda. \quad (14) \]

\[ w_1(x_1, 0) - w_2(x_2, 0) = \sqrt{\frac{2}{\pi}} \int_0^\infty \left[ A(\lambda) - C(\lambda) \right] \cos \lambda x d\lambda \quad (|x| > l) \quad (15) \]
From (15), using the Dirichlet discontinuous factor

\[
\int_0^\infty \frac{\sin \lambda t}{\lambda} \cos \lambda x d\lambda = \begin{cases} 
\pi / 2, & x < t, \\
\pi / 4, & x = t, \\
0, & x > t, 
\end{cases}
\]

we get

\[
A(\lambda) - C(\lambda) = \frac{k + 1}{\mu_1} \sqrt{\frac{2}{\pi}} \int_0^l f(t) \frac{\sin \lambda t}{\lambda} dt.
\]

Here, \(f(t)\) is a new unknown function.

And it can be proved that \(f(x) \in K_{1/2} ]-l,l[\), that is:

\[
f(x) = \frac{f_0(x)}{\sqrt{l^2 - x^2}}, \quad f_0(-x) = -f_0(x), \quad f_0(x) \in H^\beta [0,l], \quad 1/2 < \beta < 1.
\]
If condition (3) is satisfied, we reduce to a generalized singular integral equation of the first kind

\[
\sigma(x) = \frac{1}{2h_1ch^2\left(\frac{\pi x}{2h_1}\right)} \int_{-l}^{l} f(t) \left( th \frac{\pi t}{2h_1} + \frac{\pi t}{2h_1} - th^2 \frac{\pi x}{2h_1} \right) dt +
\]

\[
+ \frac{1}{\pi} \int_{-l}^{l} f(t) K(x,t) dt,
\]

\[
K(x,t) = k \int_{0}^{\infty} \frac{cth\lambda h_2 - cth\lambda h_1}{kcth\lambda h_1 + cth\lambda h_2} cth\lambda h_1 \sin \lambda t \cos \lambda x \cdot d\lambda \quad (20)
\]

\[
(|x| \leq l).
\]
Let's \( h_1 = h_2 = h \).

Then

\[
f(x) = - \frac{th \frac{\pi}{2h} x}{h \cdot ch^2 \left( \frac{\pi x}{2h} \right)} \cdot \frac{1}{\sqrt{t h^2 \frac{\pi}{2h} l - th^2 \frac{\pi}{2h} x}} \times \\
\sqrt{th^2 \frac{\pi}{2h} l - th^2 \frac{\pi}{2h} \xi} \cdot \int_{0}^{l} \sigma(\xi) ch^2 \frac{\pi}{2h} \xi \cdot d\xi.
\]

Value of the stress intensity factor \( K_{III} \)

\[
K_{III} = \frac{1}{ch \frac{\pi}{2h} l} \sqrt{\frac{2}{th \frac{\pi}{2h} l}} \cdot \int_{0}^{l} \frac{\sigma(\xi) ch^2 \frac{\pi}{2h} \xi}{\sqrt{th^2 \frac{\pi}{2h} l - th^2 \frac{\pi}{2h} \xi}} d\xi.
\]
Conclusion 1

1. If $h_1 = h_2 = h$, then the stress intensity factor $K_{III}$ does not depend on $k$, where $k = \mu_1 / \mu_2$.

2. Let $h \to +\infty$, and $h_2 = \text{const}$. Then, we reduce to the Fredholm integral equation of the second kind

$$
\frac{2}{\pi} \int_{0}^{x} \frac{\sigma(\tau)}{\sqrt{\tau^2 - x^2}} d\tau = \psi(x) + \int_{0}^{l} \psi(t) K_{\phi}(x,t) dt,
$$

$$
K_{\phi}(x,t) = k \int_{0}^{\infty} \frac{\lambda t e^{-\lambda h_2}}{ch\lambda h_2 + k \cdot sh\lambda h_2} J_0(\lambda t) J_0(\lambda x) d\lambda,
$$

$$
\psi(x) = \frac{2}{\pi} \int_{x}^{l} \frac{f(\tau)}{\sqrt{\tau^2 - x^2}} d\tau, \quad 0 \leq x \leq l, \quad f(\tau) \in K_{1/2} ]-l,l[.
$$

Here, $J_0(u)$ is a Bessel's zero-order function of the first kind, and the function $\psi(x)$ belongs to the class of continuous functions $C[0,l]$.

$$
K_{III} = \sqrt{\pi l} \cdot \psi(l),
$$

since $\psi(x) = f_0(l)/l$.

Conclusion 2. If $h_1 \neq h_2$, then the stress intensity factor $K_{III}$ depends on $k$, where $k = \mu_1 / \mu_2$. 